2020-2021 Exam Solutions

SECTION A

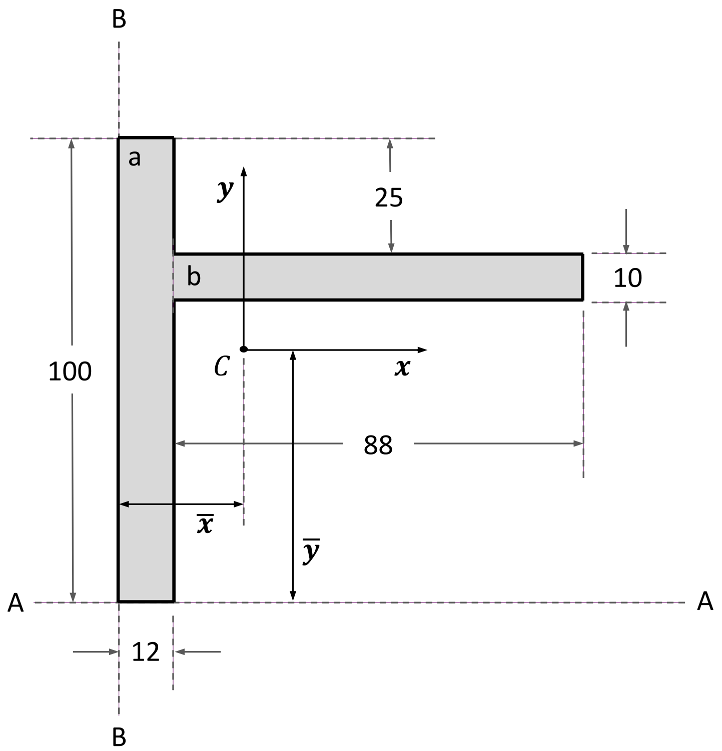
1.

C. $\bar{x} = 27.15\ \text{mm},\ \bar{y} = 58.46\ \text{mm}$

[2 marks]

SOLUTION 1

Splitting the section into convenient, rectangular sections,



Total area,

$$A = (12 \times 100)_\mathrm{a} + (88 \times 10)_\mathrm{b} = 2080\ \mathrm{mm}^4$$

Taking moments about AA:

$$\bar{y} = \frac{(12 \times 100 \times 50)_\mathrm{a} + (88 \times 10 \times 70)_\mathrm{b}}{2080}$$

$$\therefore \bar{y} = \mathbf{58.46\ mm}$$

Similarly, taking moments about BB:

$$\bar{x} = \frac{(100 \times 12 \times 6)_{\mathrm{a}} + (10 \times 88 \times 56)_{\mathrm{b}}}{2080}$$

$$\therefore \bar{x} = \mathbf{27.15\ mm}$$

2.

B. $507{,}692.32\ \mathrm{mm}^4$

[2 marks]

SOLUTION 2

$$I_{x'y'} = \left(I_{xy} + Aab\right)_{\mathrm{a}} + \left(I_{xy} + Aab\right)_{\mathrm{b}}$$

$$= \big(0 + 12 \times 100 \times (27.15 - 6) \times (58.46 - 50)\big) + \big(0 + 88 \times 10 \times (27.15 - 56) \times (58.46 - 70)\big)$$

$$= \mathbf{507,692.32\ mm^4}$$

3.

D. $u = \frac{3P^2}{EI}\left(\frac{L}{4} + \frac{3\pi L^3 R}{12} + 2\pi R^2\right)$

[2 marks]

SOLUTION 3

Deflection, u_P, at the position of and in the direction of applied load, P, is:

$$u_P = \frac{\partial U}{\partial P}$$

$$\therefore u_P = \frac{\partial\left(\frac{P^3}{EI}\left(\frac{L}{4} + \frac{3\pi L^3 R}{12} + 2\pi R^2\right) + \frac{Q^2}{EI}\left(\frac{L}{3} + \frac{3\pi R}{8}\right)\right)}{\partial P}$$

$$= \boldsymbol{\frac{3P^2}{EI}\left(\frac{L}{4} + \frac{3\pi L^3 R}{12} + 2\pi R^2\right)}$$

4.

A. $u_Q = \frac{2Q}{EI}\left(\frac{L}{3} + \frac{3\pi R}{8}\right)$

[2 marks]

SOLUTION 4

Deflection, u_Q, at the position of and in the direction of dummy load, Q, is:

$$u_Q = \frac{\partial U}{\partial P}$$

$$\therefore u_Q = \frac{\partial\left(\frac{P^3}{EI}\left(\frac{L}{4} + \frac{3\pi L^3 R}{12} + 2\pi R^2\right) + \frac{Q^2}{EI}\left(\frac{L}{3} + \frac{3\pi R}{8}\right)\right)}{\partial P}$$

$$= \boldsymbol{\frac{2Q}{EI}\left(\frac{L}{3} + \frac{3\pi R}{8}\right)}$$

5.

A. $EI\frac{d^2y}{dx^2} = M$

[2 marks]

6.

E. Increases the load required to cause buckling

[2 marks]

SOLUTION 6

All buckling load equations have length, L, on the bottom of the equation. Therefore, increasing the length of the beam reduces the load required to cause buckling. An example, for a hinged-hinged beam, is shown below.

$$P_c = \frac{\pi^2 EI}{L^2}$$

7.

D. C1-ID $\sigma_r = 0$, C1-OD $\sigma_r = -p$, C2-ID $\sigma_r = -p$, C2-OD $\sigma_r = 0$

[2 marks]

8.

E. $14.6\ \text{MPa}$

[2 marks]

SOLUTION 8

Second moment of Area:

$$I = \frac{100 \times 100^3}{12} - 2 \times \frac{40 \times 60^3}{12} = 6.83 \times 10^6\ \text{mm}^4$$

$$\tau_{max} = \frac{SB}{8I}(D + d) = \frac{50000 \times 100}{8 \times 6.83 \times 10^6}(100 + 60) = \mathbf{14.6\ MPa}$$

9.

B. $21.4\ \text{MPa}$

[2 marks]

SOLUTION 9

The maximum shear stress at the N.A. given by:

$$\tau = \frac{SA\bar{y}}{Iz} = \frac{12 \times 15000 \times (17.5 \times 30) \times 8.75}{30 \times 35^3 \times 30} = \mathbf{21.4\ MPa}$$

10.

D. $[k_{el}] = 2.52 \times 10^8 \begin{bmatrix} 1.667 & -1.667 \\ -1.667 & 1.667 \end{bmatrix}$

[2 marks]

SOLUTION 10

$$[k_{el}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

$$k = \frac{A \times E}{L} = \frac{1.2 \times 10^{-3} \times 210 \times 10^9}{0.6} = \mathbf{4.2 \times 10^8} \; (= \mathbf{2.52 \times 10^8 \times 1.667})$$

11.

B. 44 mm

[2 marks]

SOLUTION 11

For a shaft under pure torque, the Mohr's circle is centred on the origin and σ_1 and σ_3 will be the same magnitude and will also be the maximum allowable shear stress, therefore according to the Tresca yield criterion:

$$\tau_{max} = (\sigma_1 - \sigma_3)/2 = \sigma_y/2 = R$$

or

$$\frac{315}{2} = \tau_{max} = 157.5 \text{ MPa}$$

$$157.5 \times 10^6 = \frac{Tr}{J} = \frac{2T}{\pi r^3}$$

$$r = \sqrt[3]{\frac{2 \times 21000}{157.5 \times 10^6 \pi}} = \mathbf{0.044 = 44\ mm}$$

12.

C. 86.5 mm

[2 marks]

SOLUTION 12

$$K_I = 1.85\sigma\sqrt{\pi a}$$

$$\therefore K_{I_{cr}} = 1.85 \times \frac{3}{4}\sigma_y\sqrt{\pi a_{cr}} \qquad (1)$$

where

$$K_{I_{cr}} = 170 \text{ MPa}\sqrt{\text{m}}$$

and

$$\sigma_y = 235 \text{ MPa}$$

Substituting these values for $K_{I_{cr}}$ and σ_y into (1) gives:

$$170 = 1.85 \times \frac{3}{4} \times 235 \times \sqrt{\pi a_{cr}}$$

$$\therefore a_{cr} = \left(\frac{170}{1.85 \times \frac{3}{4} \times 235}\right)^2 \times \frac{1}{\pi} = 0.0865 \text{ m} = \mathbf{86.5\ mm}$$

13.

A. More conservative than linear hardening

[2 marks]

14.

B. 120.7 MPa

[2 marks]

SOLUTION 14

Given $\sigma_z = 125$ MPa, $\sigma_y = 50$ MPa and $\tau_{zy} = 30$ MPa:

$$C = \frac{\sigma_x + \sigma_y}{2} = \frac{125 + 50}{2} = 87.5 \text{ MPa}$$

and

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{z\theta}^2} = \sqrt{\left(\frac{125 - 50}{2}\right)^2 + 30^2} = 48.0 \text{ MPa}$$

Therefore,

$$\sigma_1 = C + R = 135.5 \text{ MPa}$$

$$\sigma_2 = C - R = 39.5 \text{ MPa}$$

$$\sigma_{vm} = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} = \sqrt{135.5^2 - 135.5 \times 39.5 + 39.5^2} = \mathbf{120.7\ MPa}$$

15.

E. 510.8 MPa

[2 marks]

SOLUTION 15

Torsional shear stress:

$$\tau = \frac{Tr}{J} = \frac{32 \times 750 \times 15 \times 10^{-3}}{\pi \times (30 \times 10^{-3})^4} = 141.5 \text{ MPa}$$

Axial stress:

$$\sigma_a = \frac{My}{I} = \frac{64 \times 1250 \times 15 \times 10^{-3}}{\pi \times (30 \times 10^{-3})^4} = 471.6 \text{ MPa}$$

Centre given by:

$$C = \frac{\sigma_a}{2} = \frac{471.6}{2} = 235.8 \text{ MPa}$$

Radius is given by:

$$R = \sqrt{\left(\frac{\sigma_a}{2}\right)^2 + \tau^2} = \sqrt{235.8^2 + 141.5^2} = 275 \text{ MPa}$$

Max principal stress given by:

$$\sigma_1 = C + R = 235.8 + 275 = \mathbf{510.8\ MPa}$$

16.

C. 0.013×10^{-4}

[2 marks]

SOLUTION 16

$$\sigma_r = A - \frac{B}{r^2}$$

$$\sigma_\theta = A + \frac{B}{r^2}$$

at r = 20 (ID), $\sigma_r = -14.5$ therefore:

$$-14.5 = A - \frac{B}{20^2}$$

$$-14.5 = A - 0.0025B$$

at r = 50 (OD), $\sigma_r = -5.0$ therefore:

$$-5 = A - \frac{B}{50^2}$$

$$-5 = A - 4 \times 10^{-4}B$$

therefore:

$$B = 4523.8$$

$$A = -3.1905$$

Hoop stress at the OD is given by:

$$\sigma_\theta = -3.1905 + \frac{4523.8}{2500} = -1.38\ \mathrm{MPa}$$

The strain is given by:

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) = \frac{1}{208 \times 10^3}(-1.38 - 0.33 \times -5) = 1.3 \times 10^{-6} = \mathbf{0.0130 \times 10^{-4}}$$

17.

C. R-ratio should be decreased

[2 marks]

18.

A. (i) $x = 0, y = 0$, (ii) $x = L, y = 0$ & (iii) $x = L, \frac{\mathrm{d}y}{\mathrm{d}x} = 0$

[2 marks]

19.

E. $252.6\ \mathrm{mm}$

[2 marks]

SOLUTION 19

2nd moment of area of a solid circular cross-section:

$$I = \frac{\pi D^4}{64}$$

$$\therefore D = \sqrt[4]{\frac{64I}{\pi}}$$

Substituting in the minimum required 2nd moment of area:

$$D_{min} = \sqrt[4]{\frac{64 \times 200{,}000{,}000}{\pi}} = \mathbf{252.6\ mm}$$

20.

D. 47.7 mm

[2 marks]

SOLUTION 20

von Mises yield criteria for a plane stress case

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \geq \sigma_y^2$$

For a shaft under pure torque, the Mohr's circle is centred on the origin and σ_1 and σ_2 will be the same magnitude, *k*, and also be the maximum allowable shear stress, therefore

$$3k^2 = \sigma_y^2$$

And the value of *k* will therefore be

$$k = \frac{\sigma_y}{\sqrt{3}} = \frac{275 \times 10^6}{\sqrt{3}} = 158.77 \times 10^6 \text{ Pa}$$

Therefore

$$\tau = 158.77 \times 10^6 = \frac{Tr}{J} = \frac{2T}{\pi r^3}$$

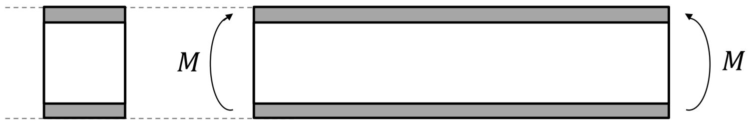
Rearranging

$$r = \sqrt[3]{\frac{2 \times 27{,}000}{158.77 \times 10^{6} \times \pi}} = 0.0477\ \mathrm{m} \approx \mathbf{47.7\ mm}$$

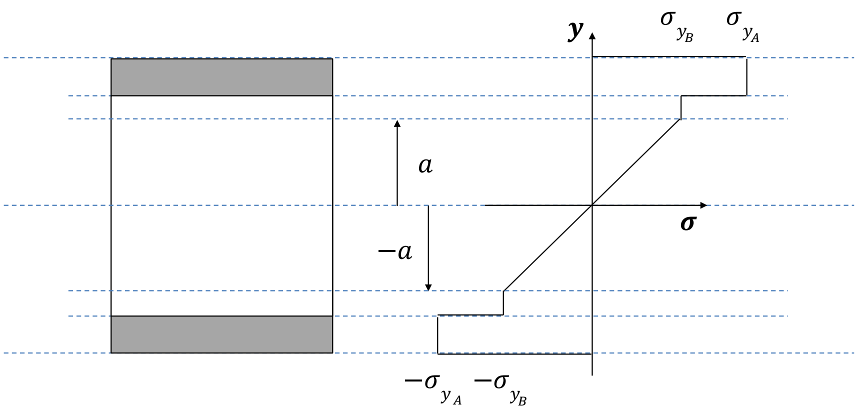
SECTION B

21.

(a)



Assuming yielding occurs at $y \geq a$ and $y \leq -a$, the stress distribution in the beam cross-section is as follows:



Cross section of the beam

Stress distribution through the beam cross section

Variation of stress with y:

- For $120 < y < 80$, $\sigma = \sigma_{y_A}$
- For $80 < y < a$, $\sigma = \sigma_{y_B}$
- For $a < y < -a$, $\sigma = \frac{\sigma_{y_B}}{a} y$
- For $-a < y < -80$, $\sigma = -\sigma_{y_B}$
- For $-120 < y < -80$, $\sigma = \sigma_{y_A}$

[3 marks]

<u>Moment equilibrium</u>

(Balance the moments due to stresses in the elastic and plastic regions with the applied moment)

$$M = \int_A y\sigma dA = \int_y y\sigma b dy$$

[1 mark]

Due to the symmetry of the stress distribution above and below the Y-Y axis, and substituting in the elastic and plastic terms for σ, this can be rewritten as:

$$M = 2\left\{\int_0^a y\frac{\sigma_{y_A}}{a}ybdy + \int_a^{80} y\sigma_{y_A}bdy + \int_{80}^{120} y\sigma_{y_B}bdy\right\}$$

[2 marks]

$$= 2b\left\{\frac{\sigma_{y_A}}{a}\int_0^a y^2dy + \sigma_{y_A}\int_a^{80} ydy + \sigma_{y_B}\int_{80}^{120} ydy\right\}$$

$$= 2b\left\{\frac{\sigma_{y_A}}{a}\left[\frac{y^3}{3}\right]_0^a + \sigma_{y_A}\left[\frac{y^2}{2}\right]_a^{80} + \sigma_{y_B}\left[\frac{y^2}{2}\right]_{80}^{120}\right\}$$

$$= 2b\left\{\frac{\sigma_{y_A}}{3a}(a^3) + \frac{\sigma_{y_A}}{2}(80^2 - a^2) + \frac{\sigma_{y_B}}{2}(120^2 - 80^2)\right\}$$

$$= 2b\left\{3200\sigma_{y_A} + 4000\sigma_{y_B} - \frac{\sigma_{y_A}a^2}{6}\right\}$$

[2 marks]

$$\therefore a = \sqrt{\frac{3}{\sigma_{y_A}}\left(9600\sigma_{y_A} + 8000\sigma_{y_B} - \frac{M}{b}\right)}$$

[2 marks]

$$= \mathbf{69.96\ mm}$$

[2 marks]

(b)

Assuming unloading to be entirely elastic. Beam bending equation:

$$\frac{M}{I} = \frac{\sigma}{y}\left(= \frac{E}{R}\right)$$

$$\therefore \frac{\Delta M}{I} = \frac{\Delta\sigma}{y}$$

[1 mark]

$$I = \left(\frac{bd^3}{12}\right) = \frac{180 \times 240^3}{12} = 207{,}360{,}000\ \text{mm}^2$$

[1 mark]

Max change in stress ($\Delta\sigma$) will occur at $y = \pm 120\ \text{mm}$ (furthest perpendicular distance from the Y-Y axis):

$$\therefore \Delta\sigma_{max}^{el} = \frac{\Delta M \times y_{max}}{I} = \frac{-M \times y_{max}}{I} = \frac{-540{,}000{,}000 \times \pm 120}{207{,}360{,}000}$$

$$= \mp 312.5 \text{ MPa}$$

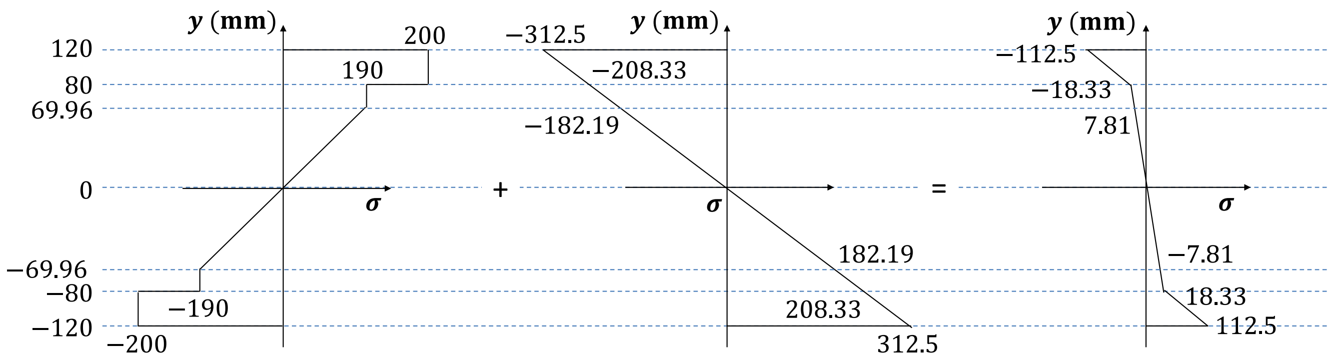
i.e.:

$$\text{at } y = 120 \text{ mm}, \therefore \Delta\sigma_{max}^{el} = -312.5 \text{ MPa}$$

$$\text{and at } y = -120 \text{ mm}, \therefore \Delta\sigma_{max}^{el} = 312.5 \text{ MPa}$$

[2 marks]

Therefore, the stress distribution after unloading can be calculated as:



Loading stress distribution

(Elastic) unloading stress distribution

Residual stress distribution

Interpolation of (elastic) unloading line:

At $y = 120$ mm, $\sigma = -312.5$ MPa
$y = m\sigma + c$
$\therefore 120 = \text{m} \times -312.5 + 0$
$\therefore \text{m} = -0.384$

At $y = 80$ mm, $80 = -0.384 \times \sigma$
$\therefore \sigma = -208.33$ MPa

At $y = 69.96$ mm, $69.96 = -0.384 \times \sigma$
$\therefore \sigma = -182.19$ MPa

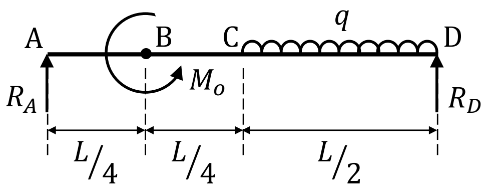
[3 marks]

The largest residual stress magnitude is 112.5 MPa, so reverse yielding does not occur, as this is below yield of the materials, and so the assumption of purely elastic unloading is reasonable.

[1 mark]

22.

Drawing a free body diagram of the beam:



[2 marks]

Taking vertical equilibrium:

$$R_A + R_D = \frac{qL}{2} \qquad \text{(i)}$$

Taking moments about position D:

$$\frac{qL^2}{8} + M_O = R_A L$$

$$\therefore R_A = \frac{qL}{8} + \frac{M_O}{L} \qquad \text{(ii)}$$

[1 mark]

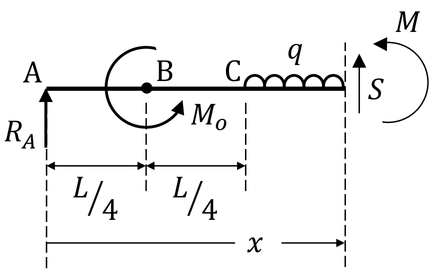
Substituting this into equation (i) gives:

$$\frac{qL}{8} + \frac{M_O}{L} + R_D = \frac{qL}{2}$$

$$\therefore R_D = \frac{qL}{2} - \frac{qL}{8} - \frac{M_O}{L} = \frac{3qL}{8} - \frac{M_O}{L}$$

[1 mark]

Take origin from right hand side and sectioning the beam after the last discontinuity:



[3 marks]

Taking moments about the section position (remembering to use Macaulay brackets where needed) gives:

$$M + \frac{q\,\langle x - \frac{L}{2}\rangle^2}{2} + M_O\,\langle x - \frac{L}{4}\rangle^0 = R_A x$$

$$\therefore M = R_A x - \frac{q\,\langle x - \frac{L}{2}\rangle^2}{2} - M_O\,\langle x - \frac{L}{4}\rangle^0$$

[2 marks]

Substituting this in the main deflection of beams equation gives:

$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{q\,\langle x - \frac{L}{2}\rangle^2}{2} - M_O\,\langle x - \frac{L}{4}\rangle^0$$

[1 mark]

Integrating gives:

$$EI\frac{dy}{dx} = \frac{R_A x^2}{2} - \frac{q\,\langle x - \frac{L}{2}\rangle^3}{6} - M_O\,\langle x - \frac{L}{4}\rangle + A \qquad \text{(iii)}$$

[1 mark]

Integrating again gives:

$$EIy = \frac{R_A x^3}{6} - \frac{q\,\langle x - \frac{L}{2}\rangle^4}{24} - \frac{M_O\,\langle x - \frac{L}{4}\rangle^2}{2} + Ax + B \qquad \text{(iv)}$$

[1 mark]

Boundary conditions:

(BC1) At $x = 0$, $y = 0$, therefore from (iv):

$$B = 0$$

[1 mark]

(BC2) At $x = L$, y = 0, therefore from (iv):

$$A = \frac{qL^3}{384} + \frac{9M_O L}{32} - \frac{R_A L^2}{6}$$

[1 mark]

Substituting this into (iii) and (iv) gives:

$$\frac{dy}{dx} = \frac{1}{EI}\left(\frac{qL^3}{384} - \frac{q\,\langle x - \frac{L}{2}\rangle^3}{6} + \frac{R_A x^2}{2} - \frac{R_A L^2}{6} + \frac{9M_O L}{32} - M_O\,\langle x - \frac{L}{4}\rangle\right) \qquad \text{(v)}$$

[1 mark]

and,

$$y = \frac{1}{EI}\left(\frac{qL^3}{384}x - \frac{q\,\langle x - \frac{L}{2}\rangle^4}{24} + \frac{R_A x^3}{6} - \frac{R_A L^2}{6}x + \frac{9M_O L}{32}x - \frac{M_O\,\langle x - \frac{L}{4}\rangle^2}{2}\right) \qquad \text{(vi)}$$

[1 mark]

At position B, $x = \frac{L}{4}$, therefore this and (ii) into (v) and (vi):

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{EI}\left(\frac{7M_O L}{48} - \frac{11qL^3}{768}\right)}$$

[1 mark]

and,

$$\boldsymbol{y = \frac{1}{EI}\left(\frac{M_O L^2}{32} - \frac{13qL^4}{3072}\right)}$$

[1 mark]

At position C, $x = \frac{L}{2}$, therefore this and (ii) into (v) and (vi):

$$\boldsymbol{\frac{dy}{dx} = -\frac{1}{EI}\left(\frac{qL^3}{384} + \frac{M_O L}{96}\right)}$$

[1 mark]

and,

$$\boldsymbol{y = \frac{1}{EI}\left(\frac{3M_O L^2}{64} - \frac{5qL^4}{768}\right)}$$

[1 mark]

23.

(a)

$$\delta_{total} = \delta_{thermal} + \delta_{mech} = 0$$

therefore:

$$0 = L\alpha\Delta T + \frac{FL}{AE}$$

rearranging:

$$F = -\alpha\Delta TAE = -11 \times 10^{-6} \times 55 \times \pi((80 \times 10^{-3})^2 - (60 \times 10^{-3})^2) \times 208 \times 10^9 = -1.1069 \times 10^6 \text{ N}$$

$$\sigma = \frac{F}{A} = \frac{-1.1069 \times 10^6}{\pi((80 \times 10^{-3})^2 - (60 \times 10^{-3})^2)} = -1.2584 \times 10^8 \text{Pa} = -125.8 \text{ MPa}$$

[2 marks]

(b)

As before:

$$\delta_{total} = \delta_{thermal} + \delta_{mech} = 0$$

where:

$$\delta_{mech} = \frac{FL}{AE}$$

However, for a small slice of the bar having a length dx, if allowed to expand freely:

$$d(\delta_T) = (\alpha\Delta T)dx$$

where:

$$\Delta T = \frac{55x}{L}$$

gives:

$$\delta_T = \int_0^L \alpha\left(\frac{55x}{L}\right)dx = \alpha\int_0^L \left(\frac{55x}{L}\right)dx$$

or:

$$\delta_T = \alpha\left(\frac{55x^2}{2L}\right)\Bigg|_0^L = \alpha\frac{55L}{2}$$

therefore:

$$0 = \alpha \frac{55L}{2} + \frac{FL}{AE}$$

and

$$F = -\alpha \frac{55AE}{2}$$

which gives:

$$F = -\alpha \frac{55AE}{2}$$

$$= -11 \times 10^{-6} \times \frac{55 \times \pi((80 \times 10^{-3})^2 - (60 \times 10^{-3})^2) \times 208 \times 10^9}{2}$$

$$= -5.5347 \times 10^5 \text{ N}$$

Therefore, the stress in the bar can be determined as:

$$\sigma = \frac{F}{A} = \frac{-5.5347 \times 10^5 \text{ N}}{\pi((80 \times 10^{-3})^2 - (60 \times 10^{-3})^2)} = -6.2920 \times 10^7 \text{Pa} = -62.92 \text{ MPa}$$

[8 marks]

(c)

BCs,

at r=60 mm, $\sigma_r = -12$ MPa

$$-12 = A - \frac{B}{3600}$$

at r=80 mm, $\sigma_r = -2.5$ MPa

$$-2.5 = A - \frac{B}{6400}$$

giving:

$$A = \frac{B}{3600} - 12$$

and

$$9.5 = B\left(\frac{1}{3600} - \frac{1}{6400}\right)$$

$$9.5 = \frac{28B}{230400}$$

Therefore:

$$B = 78171$$

and

$$A = 9.7$$

Therefore:

$$\sigma_{\theta(ro)} = A + \frac{B}{r^2} = 9.7 + \frac{78171}{80^2} = 21.91 \text{ MPa}$$

As the stresses are aligned with the principal directions:

$$\sigma_{vM}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = (\sigma_a - \sigma_r)^2 + (\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_a)^2$$

$$\sigma_{vM} = 88.98 \text{ MPa}$$

[10 marks]